13

Flow of a compressible fluid

Fluids have the capacity to change volume and density, i.e. compressibility. Gas is much more compressible than liquid.

Since liquid has low compressibility, when its motion is studied its density is normally regarded as unchangeable. However, where an extreme change in pressure occurs, such as in water hammer, compressibility is taken into account.

Gas has large compressibility but when its velocity is low compared with the sonic velocity the change in density is small and it is then treated as an incompressible fluid.

Nevertheless, when studying the atmosphere with large altitude changes, high-velocity gas flow in a pipe with large pressure difference, the drag sustained by a body moving with significant velocity in a calm gas, and the flow which accompanies combustion, etc., change of density must be taken into account.

As described later, the parameter expressing the degree of compressibility is the Mach number M. Supersonic flow, where $M > 1$, behaves very differently from subsonic flow where $M < 1$.

In this chapter, thermodynamic characteristics will be explained first, followed by the effects of sectional change in isentropic flow, flow through a convergent nozzle, and flow through a convergent–divergent nozzle. Then the adiabatic but irreversible shock wave will be explained, and finally adiabatic pipe flow with friction (Fanno flow) and pipe flow with heat transfer (Rayleigh flow).

13.1 Thermodynamical characteristics

Now, with the specific volume v and density ρ,

$$\rho v = 1 \tag{13.1}$$

A gas having the following relationship between absolute temperature T and pressure p

$$pv = RT \tag{13.2}$$

or

$$p = R\rho T \tag{13.3}$$

is called a perfect gas. Equations (13.2) and (13.3) are called its equations of state. Here R is the gas constant, and

$$R = \frac{R_0}{\mathcal{M}}$$

where R_0 is the universal gas constant ($R_0 = 8314\,\mathrm{J/(kg\,K)}$) and $\mathcal{M}$ is the molecular weight. For example, for air, assuming $\mathcal{M} = 28.96$, the gas constant is

$$R = \frac{8314}{28.96} = 287\,\mathrm{J/(kg\,K)} = 287\,\mathrm{m^2/(s^2\,K)}$$

Then, assuming internal energy and enthalpy per unit mass e and h respectively,

specific heat at constant volume: $c_v = \left(\frac{\partial e}{\partial T}\right)_v \quad \mathrm{d}e = c_v\,\mathrm{d}T$ (13.4)

Specific heat at constant pressure: $c_p = \left(\frac{\partial h}{\partial T}\right)_p \quad \mathrm{d}h = c_p\,\mathrm{d}T$ (13.5)

Here

$$h = e + pv \tag{13.6}$$

According to the first law of thermodynamics, when a quantity of heat $\mathrm{d}q$ is supplied to a system, the internal energy of the system increases by $\mathrm{d}e$, and work $p\,\mathrm{d}v$ is done by the system. In other words,

$$\mathrm{d}q = \mathrm{d}e + p\,\mathrm{d}v \tag{13.7}$$

From the equation of state (13.2),

$$p\,\mathrm{d}v + v\,\mathrm{d}p = R\,\mathrm{d}T \tag{13.8}$$

From eqn (13.6),

$$\mathrm{d}h = \mathrm{d}e + p\,\mathrm{d}v + v\,\mathrm{d}p \tag{13.9}$$

Now, since $\mathrm{d}p = 0$ in the case of constant pressure change, eqns (13.8) and (13.9) become

$$p\,\mathrm{d}v = R\,\mathrm{d}T \tag{13.10}$$

$$\mathrm{d}h = \mathrm{d}e + p\,\mathrm{d}v = \mathrm{d}q \tag{13.11}$$

Substitute eqns (13.4), (13.5), (13.10) and (13.11) into (13.7),

$$c_p\,\mathrm{d}T = c_v\,\mathrm{d}T + R\,\mathrm{d}T$$

which becomes

$$c_p - c_v = R \tag{13.12}$$

Now, $c_p/c_v = k$ (k: ratio of specific heats (isentropic index)), so

$$c_p = \frac{k}{k-1}R \tag{13.13}$$

$$c_v = \frac{1}{k-1}R \tag{13.14}$$

Whenever heat energy $\mathrm{d}q$ is supplied to a substance of absolute temperature T, the change in entropy $\mathrm{d}s$ of the substance is defined by the following equation:

$$\mathrm{d}s = \mathrm{d}q/T \tag{13.15}$$

As is clear from this equation, if a substance is heated the entropy increases, while if it is cooled the entropy decreases. Also, the higher the gas temperature, the greater the added quantity of heat for the small entropy increase.

Rewrite eqn (13.15) using eqns (13.1), (13.2), (13.12) and (13.13), and the following equation is obtained:[1]

$$\frac{\mathrm{d}q}{T} = c_v\,\mathrm{d}(\log pv^k) \tag{13.16}$$

When changing from state (p_1, v_1) to state (p_2, v_2), if reversible, the change in entropy is as follows from eqns (13.15) and (13.16):

$$s_2 - s_1 = c_v \log\left(\frac{p_2 v_2^k}{p_1 v_1^k}\right) \tag{13.17}$$

In addition, the relationships of eqns (13.18)–(13.20) are also obtained.[2]

[1] From

$$pv = RT \quad \frac{\mathrm{d}p}{p} + \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}T}{T}$$

Therefore

$$\frac{\mathrm{d}q}{T} = c_v\frac{\mathrm{d}T}{T} + \frac{p}{T}\mathrm{d}v = c_v\frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}v}{v} = c_v\frac{\mathrm{d}p}{p} + c_p\frac{\mathrm{d}v}{v} = c_v\left(\frac{\mathrm{d}p}{p} + k\frac{\mathrm{d}v}{v}\right)$$

[2] Equations (13.18), (13.19) and (13.20) are respectively induced from the following equations:

$$\mathrm{d}s = \frac{\mathrm{d}q}{T} = c_v\frac{\mathrm{d}T}{T} - R\frac{\mathrm{d}\rho}{\rho} = c_v\frac{\mathrm{d}T}{T} - (k-1)c_v\frac{\mathrm{d}\rho}{\rho}$$

$$\mathrm{d}s = \frac{\mathrm{d}q}{T} = c_v\frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}v}{v} = (c_v - R)\frac{\mathrm{d}T}{T} + R\frac{\mathrm{d}v}{v} = kc_v\frac{\mathrm{d}T}{T} - (k-1)c_v\frac{\mathrm{d}p}{p}$$

$$\mathrm{d}s = \frac{\mathrm{d}q}{T} = c_v\frac{\mathrm{d}p}{p} + c_p\frac{\mathrm{d}v}{v} = c_v\frac{\mathrm{d}p}{p} - c_p\frac{\mathrm{d}\rho}{\rho} = c_v\frac{\mathrm{d}p}{p} - kc_v\frac{\mathrm{d}\rho}{\rho}$$

$$s_2 - s_1 = c_v \log\left[\frac{T_2}{T_1}\left(\frac{\rho_1}{\rho_2}\right)^{k-1}\right] \tag{13.18}$$

$$s_2 - s_1 = c_v \log\left[\left(\frac{T_2}{T_1}\right)^k \left(\frac{p_1}{p_2}\right)^{k-1}\right] \tag{13.19}$$

$$s_2 - s_1 = c_v \log\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^k\right] \tag{13.20}$$

for the reversible adiabatic (isentropic) change, $ds = 0$. Putting the proportional constant equal to c, eqn (13.17) gives (13.21), or eqn (13.22) from (13.20). That is,

$$pv^k = c \tag{13.21}$$

$$p = c\rho^k \tag{13.22}$$

Equations (13.18) and (13.19) give the following equation:

$$T = c\rho^{k-1} = cp^{(k-1)/k} \tag{13.23}$$

When a quantity of heat ΔQ transfers from a high-temperature gas at T_1 to a low-temperature gas at T_2, the changes in entropy of the respective gases are $-\Delta Q/T_1$ and $\Delta Q/T_2$. Also, the value of their sum is never negative.[3] Using entropy, the second law of thermodynamics could be expressed as 'Although the grand total of entropies in a closed system does not change if a reversible change develops therein, it increases if any irreversible change develops.' This is expressed by the following equation:

$$ds \geq 0 \tag{13.24}$$

Consequently, it can also be said that 'entropy in nature increases'.

13.2 Sonic velocity

It is well known that when a minute disturbance develops in a gas, the resulting change in pressure propagates in all directions as a compression wave (longitudinal wave, pressure wave), which we feel as a sound. Its propagation velocity is called the sonic velocity.

Here, for the sake of simplicity, assume a plane wave in a stationary fluid in a tube of uniform cross-sectional area A as shown in Fig. 13.1. Assume that, due to a disturbance, the velocity, pressure and density increase by u, dp and $d\rho$ respectively. Between the wavefront which has advanced at sonic velocity a and the starting plane is a section of length l where the pressure has increased. Since the wave travel time, during which the pressure increases in this section, is $t = l/a$, the mass in this section increases by $Al\,d\rho/t = Aa\,d\rho$

[3] In a reversible change where an ideal case is assumed, the heat shifts between gases of equal temperature. Therefore, $ds = 0$.

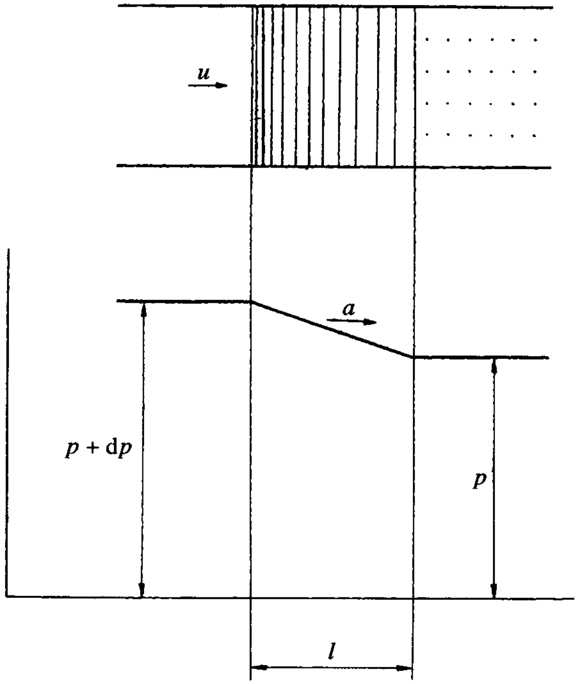


Fig. 13.1 Propagation of pressure wave

per unit time. In order to supplement it, gas of mass $Au(\rho + \mathrm{d}\rho) = Au\rho$ flows in through the left plane. In other words, the continuity equation in this case is

$$Aa\,\mathrm{d}\rho = Au\rho$$

or

$$a\,\mathrm{d}\rho = u\rho \tag{13.25}$$

The fluid velocity in this section changes from 0 to u in time t. In other words, the velocity can be regarded as having uniform acceleration $u/t = ua/l$. Taking its mass as $Al\rho$ and neglecting $\mathrm{d}\rho$ in comparison with ρ, the equation of motion is

$$Al\rho\frac{ua}{l} = A\,\mathrm{d}p$$

or

$$\rho au = \mathrm{d}p \tag{13.26}$$

Eliminate u in eqns (13.25) and (13.26), and

$$a = \sqrt{\mathrm{d}p/\mathrm{d}\rho} \tag{13.27}$$

is obtained.

Since a sudden change in pressure is regarded as adiabatic, the following equation is obtained from eqns (13.3) and (13.23):[4]

$$a = \sqrt{kRT} \tag{13.28}$$

In other words, the sonic velocity is proportional to the square root of absolute temperature. For example, for $k = 1.4$ and $R = 287\,\mathrm{m}^2/(\mathrm{s}^2\,\mathrm{K})$,

$$a = 20\sqrt{T} \quad (a = 340\,\mathrm{m/s} \text{ at } 16°\mathrm{C}\ (289\,\mathrm{K})) \tag{13.29}$$

Next, if the bulk modulus of fluid is K, from eqns (2.13) and (2.15),

$$\mathrm{d}p = -K\frac{\mathrm{d}v}{v} = K\frac{\mathrm{d}\rho}{\rho}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{K}{\rho}$$

Therefore, eqn (13.27) can also be expressed as follows:

$$a = \sqrt{K/\rho} \tag{13.30}$$

13.3 Mach number

The ratio of flow velocity u to sonic velocity a, i.e. $M = u/a$, is called Mach number (see Section 10.4.1). Now, consider a body placed in a uniform flow of velocity u. At the stagnation point, the pressure increases by $\Delta p = \rho U^2/2$ in approximation of eqn (9.1). This increased pressure brings about an increased density $\Delta\rho = \Delta p/a^2$ from eqn (13.27). Consequently,

$$M = \frac{U}{a} = \frac{1}{a}\sqrt{\frac{2\Delta p}{\rho}} = \sqrt{\frac{2\Delta\rho}{\rho}} \tag{13.31}$$

In other words, the Mach number is a non-dimensional number expressing the compressive effect on the fluid. From this equation, the Mach number M corresponding to a density change of 5% is approximately 0.3. For this reason steady flow can be treated as incompressible flow up to around Mach number 0.3.

Now, consider the propagation of a sonic wave. This minute change in pressure, like a sound, propagates at sonic velocity a from the sonic source in all directions as shown in Fig. 13.2(a). A succession of sonic waves is produced cyclically from a sonic source placed in a parallel flow of velocity u. When u is smaller than a, as shown in Fig. 13.2(b), i.e. if $M < 1$, the wavefronts propagate at velocity $a - u$ upstream but at velocity $a + u$ downstream. Consequently, the interval between the wavefronts is dense upstream while being sparse

[4] $p = c\rho^k$, $\mathrm{d}p/\mathrm{d}\rho = ck\rho^{k-1} = kp/\rho = kRT$.

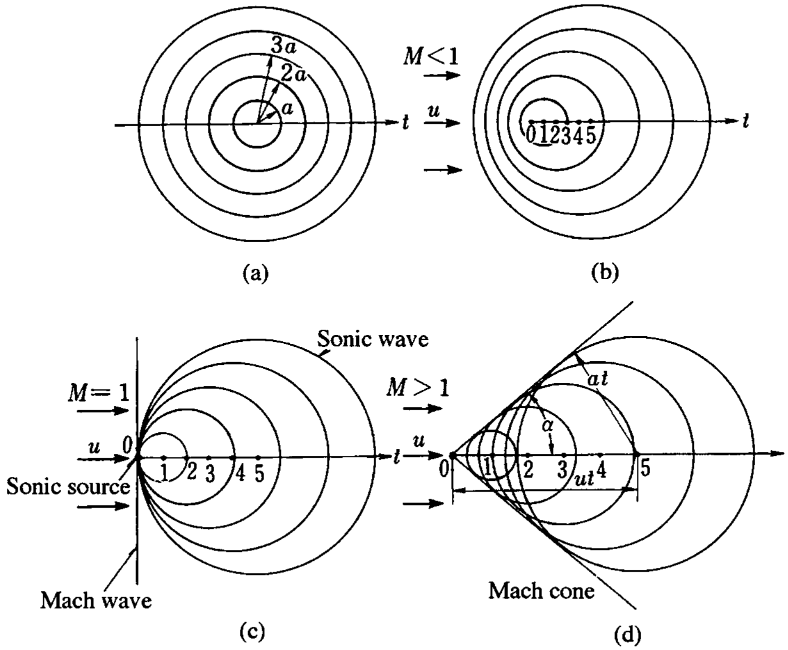


Fig. 13.2 Mach number and propagation range of a sonic wave: (a) calm; (b) subsonic ($M < 1$); (c) sonic ($M = 1$); (d) supersonic ($M > 1$)

downstream. When the upstream wavefronts therefore develop a higher frequency tone than those downstream this produces the Doppler effect.

When $u = a$, i.e. $M = 1$, the propagation velocity is just zero with the sound propagating downstream only. The wavefront is now as shown in Fig. 13.2(c), producing a Mach wave normal to the flow direction.

When $u > a$, i.e. $M > 1$, the wavefronts are quite unable to propagate upstream as in Fig. 13.2(d), but flow downstream one after another. The envelope of these wavefronts forms a Mach cone. The propagation of sound is limited to the inside of the cone only. If the included angle of this Mach cone is 2α, then[5]

$$\sin\alpha = a/u = 1/M \tag{13.32}$$

is called the Mach angle.

13.4 Basic equations for one-dimensional compressible flow

For a constant mass flow m of fluid density ρ flowing at velocity u through section area A, the continuity equation is

[5] Actually, the three-dimensional Mach line forms a cone, and the Mach angle is equal to its semi-angle.

$$m = \rho u A = \text{constant} \tag{13.33}$$

or by logarithmic differentiation

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{13.34}$$

Euler's equation of motion in the steady state along a streamline is

$$\frac{1}{\rho}\frac{dp}{ds} + \frac{d}{ds}\left(\frac{1}{2}u^2\right) = 0$$

or

$$\int \frac{dp}{\rho} + \frac{1}{2}u^2 = \text{constant} \tag{13.35}$$

Assuming adiabatic conditions from $p = c\rho^k$,

$$\int \frac{dp}{\rho} = \int ck\rho^{k-2}\, d\rho = \frac{k}{k-1}\frac{p}{\rho} + \text{constant}$$

Substituting into eqn (13.35),

$$\frac{k}{k-1}\frac{p}{\rho} + \frac{1}{2}u^2 = \text{constant} \tag{13.36}$$

or

$$\frac{k}{k-1}RT + \frac{1}{2}u^2 = \text{constant} \tag{13.37}$$

Equations (13.36) and (13.37) correspond to Bernoulli's equation for an incompressible fluid.

If fluid discharges from a very large vessel, $u = u_o \approx 0$ (using subscript 0 for the state variables in the vessel), eqn (13.37) gives

$$\frac{k}{k-1}RT + \frac{1}{2}u^2 = \frac{k}{k-1}RT_0$$

or

$$\frac{T_0}{T} = 1 + \frac{1}{RT}\frac{k-1}{k}\frac{u^2}{2} = 1 + \frac{k-1}{2}M^2 \tag{13.38}$$

In this equation, T_0, T and $\frac{1}{R}\frac{k-1}{k}\frac{u^2}{2}$ are respectively called the total temperature, the static temperature and the dynamic temperature.

From eqns (13.23) and (13.38),

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{k/(k-1)} = \left(1 + \frac{k-1}{2}M^2\right)^{k/(k-1)} \tag{13.39}$$

This is applicable to a body placed in the flow, e.g. between the stagnation point of a Pitot tube and the main flow.

Correction to a Pitot tube (see Section 11.1.1)

Putting p_∞ as the pressure at a point not affected by a body and making a binomial expansion of eqn (13.39), then (in the case where $M < 1$)

Table 13.1 Pitot tube correction

M	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$(p_0 - p_\infty)/\frac{1}{2}\rho u^2 = c$	1.000	1.003	1.010	1.023	1.041	1.064	1.093	1.129	1.170
Relative error of $u = (\sqrt{c} - 1) \times 100\%$	0	0.15	0.50	1.14	2.03	3.15	4.55	6.25	8.17

$$p_0 = p_\infty\left(1 + \frac{k}{2}M^2 + \frac{k}{8}M^4 + \frac{k(2-k)}{48}M^6 + \Lambda\right)$$
$$= p_\infty + \frac{1}{2}\rho u^2\left(1 + \frac{1}{4}M^2 + \frac{2-k}{24}M^4 + \Lambda\right) \qquad (13.40)[6]$$

For an incompressible fluid, $p_0 = p_\infty + \frac{1}{2}\rho u^2$. Consequently, for the case when the compressibility of fluid is taken into account, the correction appearing in Table 13.1 is necessary.

From Table 13.1, it is found that, when $M = 0.7$, the true flow velocity is approximately 6% less than if the fluid was considered to be incompressible.

13.5 Isentropic flow

13.5.1 Flow in a pipe (Effect of sectional change)

Consider the flow in a pipe with a gradual sectional change, as shown in Fig. 13.3, having its properties constant across any section. For the fluid at sections 1 and 2 in Fig. 13.3,

continuity equation: $$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A} = 0 \qquad (13.41)$$

equation of momentum conservation: $$-\,\mathrm{d}p\,A = (A\rho u)\mathrm{d}u \qquad (13.42)$$

isentropic relationship: $$p = c\rho^k \qquad (13.43)$$

sonic velocity: $$a^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \qquad (13.44)$$

From eqns (13.41), (13.42) and (13.44),

$$-\,a^2\,\mathrm{d}\rho = \rho u\,\mathrm{d}u = \rho u^2\frac{\mathrm{d}u}{u}$$
$$M^2\frac{\mathrm{d}u}{u} = -\frac{\mathrm{d}\rho}{\rho} = \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}A}{A}$$

6

$$p_\infty k M^2 = p_\infty k\frac{u^2}{a^2} = p_\infty\frac{ku^2}{kRT} = \frac{p_\infty}{RT}u^2 = \rho u^2$$

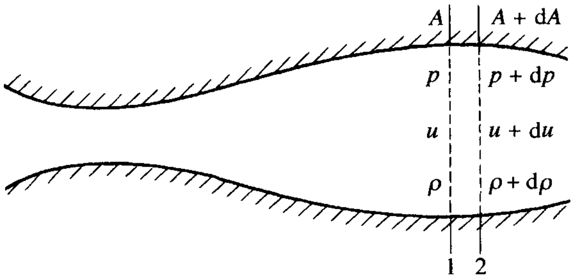


Fig. 13.3 Flow in pipe with gentle sectional change

Therefore

$$(M^2 - 1)\frac{du}{u} = \frac{dA}{A} \tag{13.45}$$

or

$$\frac{du}{dA} = \frac{1}{M^2 - 1}\frac{u}{A} \tag{13.46}$$

Also,

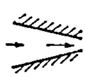
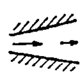
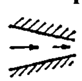
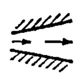
$$\frac{d\rho}{\rho} = -M^2\frac{du}{u} \tag{13.47}$$

Therefore,

$$-\frac{d\rho}{\rho}\bigg/\frac{du}{u} = M^2 \tag{13.48}$$

From eqn (13.46), when $M < 1$, $du/dA < 0$, i.e. the flow velocity decreases with increased sectional area, but when $M > 1$, $-d\rho/\rho > du/u$, i.e. for supersonic flow the density decreases at a faster rate than the velocity increases. Consequently, for mass continuity, the surprising fact emerges that in order to increase the flow velocity the section area should increase rather than decrease, as for subsonic flow.

Table 13.2 Subsonic flow and supersonic flow in one-dimensional isentropic flow

	Flow state			
	Subsonic		Supersonic	
Changing item				
Changing area	−	+	−	+
Changing velocity/Mach number	+	−	−	+
Changing density/pressure/temperature	−	+	+	−

From eqn (13.47), the change in density is in reverse relationship to the velocity. Also from eqn (13.23), the pressure and the temperature change in a similar manner to the density. The above results are summarised in Table 13.2.

13.5.2 Convergent nozzle

Gas of pressure p_0, density ρ_0 and temperature T_0 flows from a large vessel through a convergent nozzle into the open air of back pressure p_b isentropically at velocity u, as shown in Fig. 13.4. Putting p as the outer plane pressure, from eqn (13.36)

$$\frac{u^2}{2} + \frac{k}{k-1}\frac{p}{\rho} = \frac{k}{k-1}\frac{p_0}{\rho_0}$$

Using eqn (13.23) with the above equation,

$$u = \sqrt{2\frac{k}{k-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{(k-1)/k}\right]} \qquad (13.49)$$

Therefore, the flow rate is

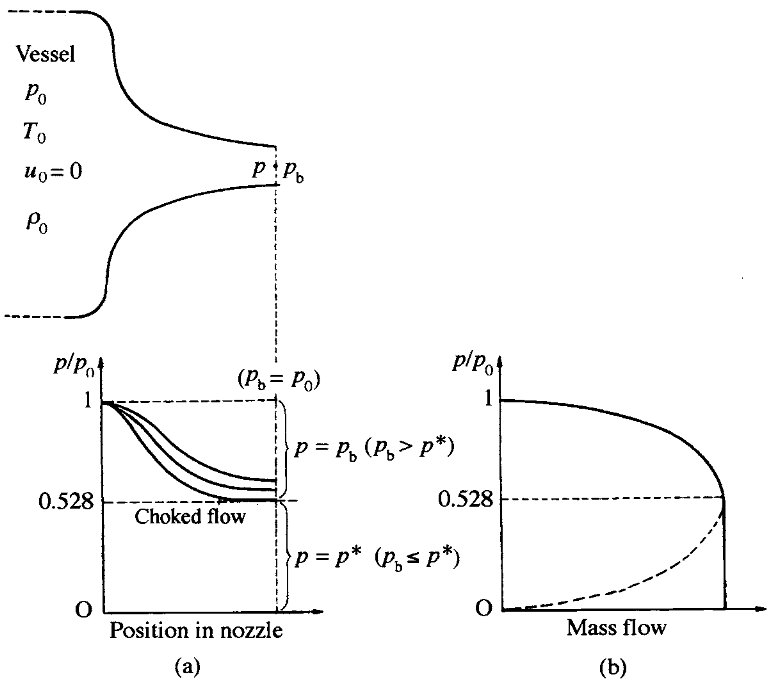


Fig. 13.4 Flow passing through convergent nozzle

$$m = \rho u A = A\sqrt{2\frac{k}{k-1}p_0\rho_0\left(\frac{p}{p_0}\right)^{2/k}\left[1-\left(\frac{p}{p_0}\right)^{(k-1)/k}\right]} \tag{13.50}$$

Writing $p/p_0 = x$, then

$$\frac{\partial m}{\partial x} = 0 \quad x = \frac{p}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \tag{13.51}$$

When p/p_0 has the value of eqn (13.51), m is maximum. The corresponding pressure is called the critical pressure and is written as p^*. For air,

$$p^*/p_0 = 0.528 \tag{13.52}$$

Using the relationship between m and p/p_0 in eqn (13.50), the maximum flow rate occurs when $p/p_0 = 0.528$ as shown in Fig 13.4(b). Thereafter, however much the pressure p_b downstream is lowered, the pressure there cannot propagate towards the nozzle because it is discharging at sonic velocity. Therefore, the pressure of the air in the outlet plane remains p^*, and the mass flow rate does not change. In this state the flow is called choked.

Substitute eqn (13.51) into (13.49) and use the relationship $p_0/\rho_0^k = p/\rho^k$ to obtain

$$u^* = \sqrt{k\frac{p}{\rho}} = a \tag{13.53}$$

In other words, for $M = 1$, under these conditions u is called the critical velocity and is written as u^*. At the same time

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)} = 0.634 \tag{13.54}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1} = 0.833 \tag{13.55}$$

The relationships of the above equations (13.52), (13.54) and (13.55) show that, at the critical outlet state $M = 1$, the critical pressure falls to 52.5% of the pressure in the vessel, while the critical density and the critical temperature respectively decrease by 37% and 17% from those of the vessel.

13.5.3 Convergent–divergent nozzle

A convergent–divergent nozzle (also called the de Laval nozzle) is, as shown in Fig. 13.5,[7] a convergent nozzle followed by a divergent length. When back pressure p_b outside the nozzle is reduced below p_0, flow is established. So long as the fluid flows out through the throat section without reaching the critical pressure the general behaviour is the same as for incompressible fluid.

When the back pressure decreases further, the pressure at the throat section

[7] Liepmann, H. W. and Roshko, A., *Elements of Gasdynamics*, (1975), 127, John Wiley, New York.

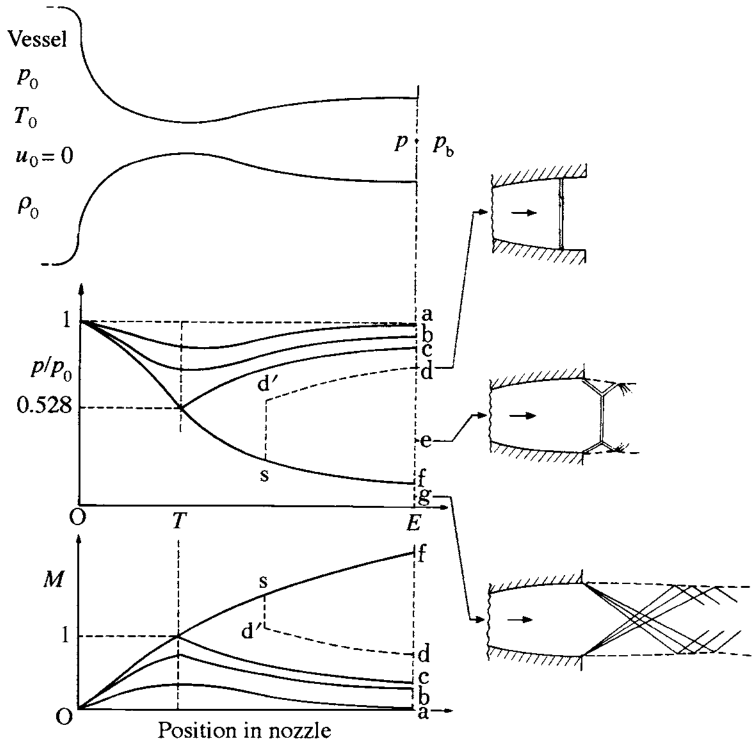


Fig. 13.5 Compressive fluid flow passing through divergent nozzle

reaches the critical pressure and $M = 1$; thereafter the flow in the divergent port is at least initially supersonic. However, unless the back pressure is low enough, supersonic velocity cannot be maintained. Instead, a shock wave develops, after which the flow becomes subsonic. As the back pressure is replaced, the shock moves further away from the diverging length to the exit plane and eventually disappears, giving a perfect expansion.

A real ratio A/A^* between the outlet section and the throat giving this perfect expansion is called the area ratio, and, using eqns (13.50) and (13.51),

$$\frac{A}{A^*} = \left(\frac{2}{k+1}\right)^{1/(k-1)} \left(\frac{p_0}{p}\right)^{1/k} \Bigg/ \sqrt{\frac{k+1}{k-1}\left[1 - \left(\frac{p_0}{p}\right)^{(1-k)/k}\right]} \tag{13.56}$$

13.6 Shock waves

When air undergoes large and rapid compression (e.g. following an explosion, the release of engine gases into an exhaust pipe, or where an aircraft or a bullet flies at supersonic velocity) a thin wave of large pressure

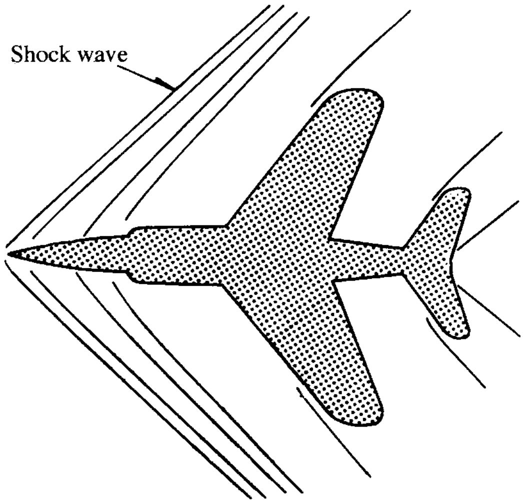


Fig. 13.6 Jet plane flying at supersonic velocity

change is produced as shown in Figs 13.6 and 13.7. Since the state of gas changes adiabatically, an increased temperature accompanies this increased pressure. As shown in Fig. 13.8(a), the wave face at the rear of the compression wave, being at a higher temperature, propagates faster than the wave face at the front. The rear therefore gradually catches up with the front until finally, as shown in Fig. 13.8(b), the wave faces combine into a thin wave increasing the pressure discontinuously. Such a pressure discontinuity is called a shock wave, which is only associated with an increase, rather than a reduction, in pressure in the flow direction.

Since a shock wave is essentially different from a sound wave because of the large change in pressure, the propagation velocity of the shock is larger, and the larger the pressure rise, the greater the propagation velocity.

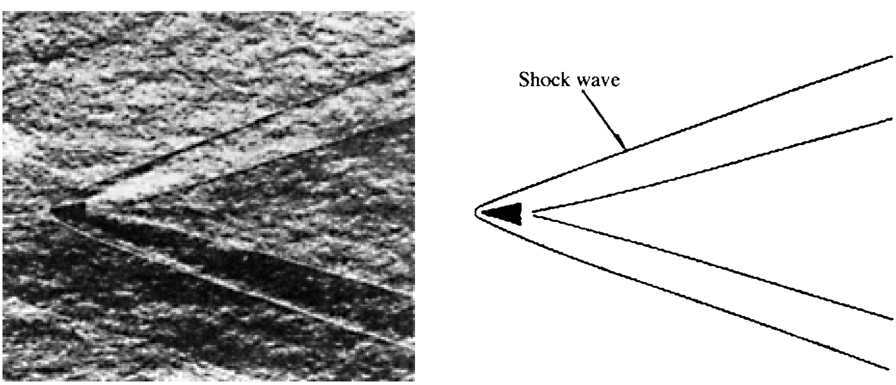


Fig. 13.7 Cone flying at supersonic velocity (Schlieren method) in air, Mach 3

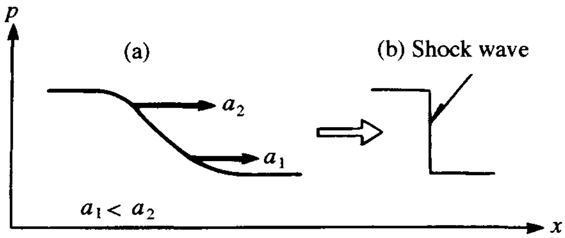


Fig. 13.8 Propagation of a compression wave

If a long cylinder is partitioned with Cellophane film or aluminium foil to give a pressure difference between the two sections, and then the partition is ruptured, a shock wave develops. The shock wave in this case is at right angles to the flow, and is called a normal shock wave. The device itself is called a shock tube.

As shown in Fig. 13.9, the states upstream and downstream of the shock wave are respectively represented by subscripts 1 and 2. A shock wave Δx is so thin, approximately micrometres at thickest, that it is normally regarded as having no thickness.

Now, assuming $A_1 = A_2$, the continuity equation is

$$\rho_1 u_1 = \rho_2 u_2 \tag{13.57}$$

the equation of momentum conservation is

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{13.58}$$

and the equation of energy conservation is

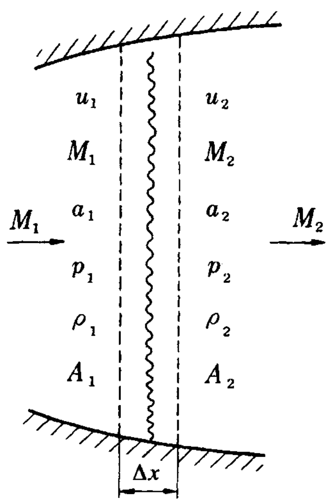


Fig. 13.9 Normal shock wave

$$\frac{u_1^2}{2}+\frac{k}{k-1}\frac{p_1}{\rho_1}=\frac{u_2^2}{2}+\frac{k}{k-1}\frac{p_2}{\rho_2}$$

or

$$u_1^2-u_2^2=\frac{2k}{k-1}\left(\frac{p_2}{\rho_2}-\frac{p_1}{\rho_1}\right) \tag{13.59}$$

From eqns (13.57) and (13.58),

$$u_1^2=\frac{p_2-p_1}{\rho_2-\rho_1}\frac{\rho_2}{\rho_1} \tag{13.60}$$

$$u_2^2=\frac{p_2-p_1}{\rho_2-\rho_1}\frac{\rho_1}{\rho_2} \tag{13.61}$$

Substituting eqns (13.60) and (13.61) into (13.59),

$$\frac{\rho_2}{\rho_1}=\frac{[(k+1)/(k-1)](p_2/p_1)+1}{[(k+1)/(k-1)]+p_2/p_1}=\frac{u_1}{u_2} \tag{13.62}$$

Or, using eqn (13.3),

$$\frac{T_2}{T_1}=\frac{[(k+1)/(k-1)]+p_2/p_1}{[(k+1)/(k-1)]+p_1/p_2} \tag{13.63}$$

Equations (13.62) and (13.63), which are called the Rankine–Hugoniot equations, show the relationships between the pressure, density and temperature ahead of and behind a shock wave. From the change of entropy associated with these equations it can be deduced that a shock wave develops only when the upstream flow is supersonic.[8]

It has already been explained that when a supersonic flow strikes a particle, a Mach line develops. On the other hand, when a supersonic flow flows along a plane wall, numerous parallel Mach lines develop as shown in Fig. 13.10(a).

When supersonic flow expands around a curved wall as shown in Fig. 13.10(b), the Mach waves rotate, forming an expansion 'fan'. This flow is called a Prandtl–Meyer expansion.

In Fig. 13.10(c), a compressive supersonic flow develops where numerous Mach lines change their direction, converging and overlapping to develop a sharp change of pressure and density, i.e. a shock wave.

[8] From eqns (13.57) and (13.58),

$$\frac{p_2}{p_1}=1+\frac{2k}{k+1}(M_1^2-1)$$

Likewise

$$\frac{p_1}{p_2}=1+\frac{2k}{k+1}(M_2^2-1)$$

Therefore

$$M_2^2=\frac{2+(k-1)M_1^2}{2kM_1^2-(k-1)}$$

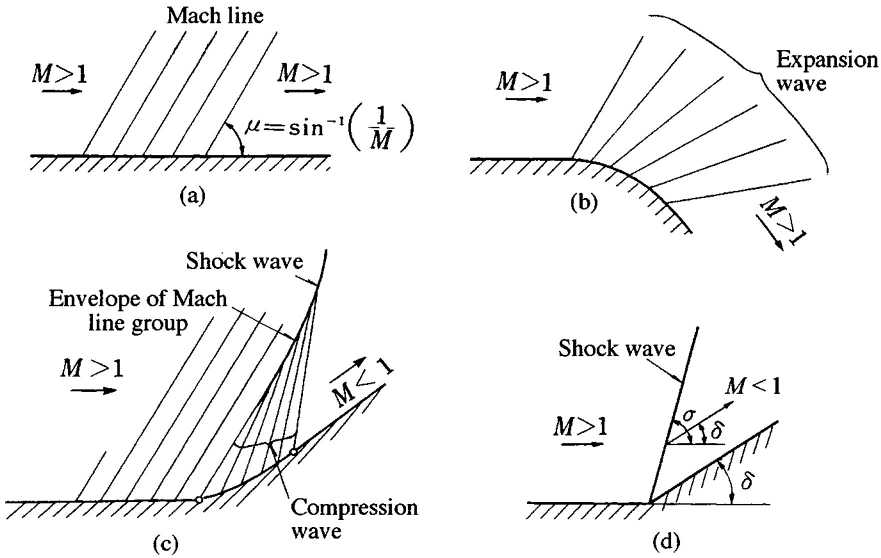


Fig. 13.10 Supersonic flow along various wave shapes

Figure 13.10(d) shows the ultimate state of a shock wave due to supersonic flow passing along this concave wall. Here, δ is the deflection angle and σ is the shock wave angle.

A shock wave is called a normal shock wave when $\sigma = 90°$ and an oblique shock wave in other cases.

From Fig. 13.11, the following relationships arise between the normal component u_n and the tangential component u_t of the flow velocity through an oblique shock wave:

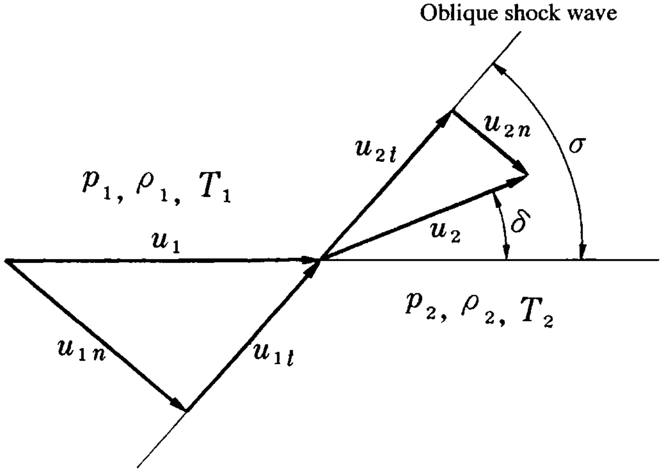


Fig. 13.11 Velocity distribution in front of and behind an oblique shock wave

$$\begin{aligned} u_{1n} &= u_1 \sin\sigma & u_{1t} &= u_1 \cos\sigma \\ u_{2n} &= u_2 \sin(\sigma-\delta) & u_{2t} &= u_2 \cos(\sigma-\delta) \end{aligned} \quad (13.64)$$

From the momentum equation in the tangential direction, since there is no pressure gradient,

$$u_{1t} = u_{2t} \quad (13.65)$$

From the momentum equation in the normal direction,

$$u_{1n}^2 - u_{2n}^2 = \frac{2k}{k-1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) \quad (13.66)$$

This equation is in the same form as eqn (13.59), and the Rankine–Hugoniot equations apply. When combined with eqn (13.64), the following relationship is developed between δ and σ:

$$\cos\delta = \left(\frac{k+1}{2}\frac{M_1^2}{M_1^2\sin^2\sigma - 1} - 1\right)\tan\sigma \quad (13.67)$$

When the shock angle $\sigma = 90°$ and $\sigma = \sin^{-1}(1/M_1)$, $\delta = 0$ so the maximum value δ_m of δ must lie between these values.

The shock wave in the case of a body where $\delta < \delta_m$ (Fig. 13.12(a)) is attached to the sharp nose A. In the case of a body where $\delta > \delta_m$ (Fig. 13.12(b)), however, the shock wave detaches and stands off from nose A.

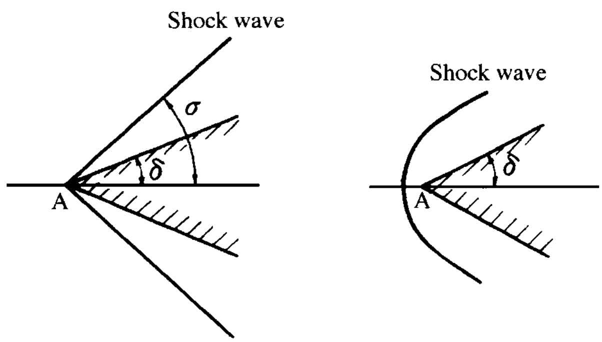


Fig. 13.12 Flow pattern and shock wave around body placed in supersonic flow: (a) shock wave attached to wedge; (b) detached shock wave

13.7 Fanno flow and Rayleigh flow

Since an actual flow of compressible fluid in pipe lines and similar conduits is always affected by the friction between the fixed wall and the fluid, it can be adiabatic but not isentropic. Such an adiabatic but irreversible (i.e. non-isentropic) flow is called Fanno flow.

Alternatively, in a system of flow forming a heat exchanger or combustion process, friction may be neglected but transfer of heat must be taken into

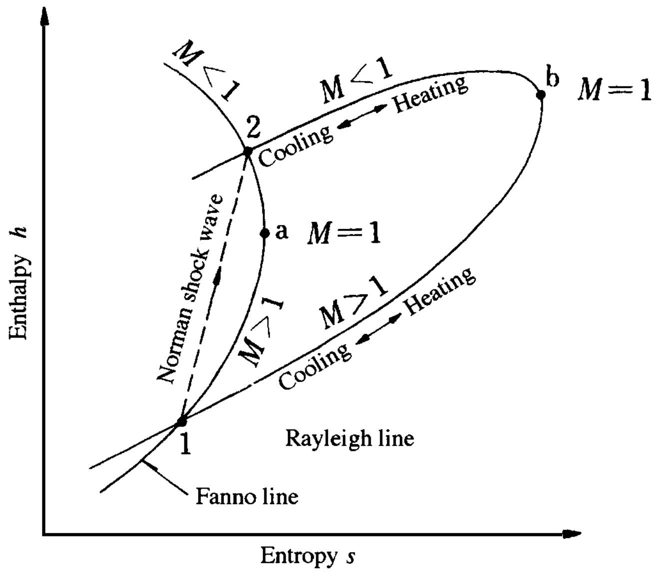


Fig. 13.13 Fanno line and Rayleigh line

account. Such a flow without friction through a pipe with heat transmission is called Rayleigh flow.

Figure 13.13 shows a diagram of both of these flows in a pipe with fixed section area. The lines appearing there are called the Fanno line and Rayleigh line respectively. For both of them, points a or b of maximum entropy correspond to the sonic state $M = 1$. The curve above these points corresponds to subsonic velocity and that below to supersonic velocity.

The states immediately ahead of and behind the normal shock wave are expressed by the intersection points 1 and 2 of these two curves. For the flow through the shock wave, only the direction of increased entropy, i.e. the discontinuous change, $1 \rightarrow 2$ is possible.

13.8 Problems

1. When air is regarded as a perfect gas, what is the density in kg/m^3 of air at 15°C and 760 mm Hg?
2. Find the velocity of sound propagating in hydrogen at 16°C.
3. When the velocity is 30 m/s, pressure 3.5×10^5 Pa and temperature 150°C at a point on a streamline in an isentropic air flow, obtain the pressure and temperature at the point on the same streamline of velocity 100 m/s.
4. Find the temperature, pressure and density at the front edge (stagnation point) of a wing of an aircraft flying at 900 km/h in calm air of pressure 4.5×10^4 Pa and temperature −26°C.

5. From a Schlieren photograph of a small bullet flying in air at 15°C and standard atmospheric pressure, it was noticed that the Mach angle was 50°. Find the velocity of this bullet.

6. When a Pitot tube was inserted into an air flow at high velocity, the pressure at the stagnation point was 1×10^5 Pa, the static pressure was 7×10^4 Pa, and the air temperature was −10°C. Find the velocity of this air flow.

7. Air of gauge pressure 6×10^4 Pa and temperature 20°C is stored in a large tank. When this air is released through a convergent nozzle into air of 760 mm Hg, find the flow velocity at the nozzle exit.

8. Air of gauge pressure 1.2×10^5 Pa and temperature 15°C is stored in a large tank. When this air is released through a convergent nozzle of exit area 3 cm^2 into air of 760 mm Hg, what is the mass flow?

9. Find the divergence ratio necessary for perfectly expanding air under standard conditions down to 100 mm Hg absolute pressure through a convergent–divergent nozzle.

10. The nozzle for propelling a rocket is a convergent–divergent nozzle of throat cross-sectional area 500 cm^2. Regard the combustion gas as a perfect gas of mean molecular weight 25.8 and $\kappa = 1.25$. In order to make the combustion gas of pressure 32×10^5 Pa and temperature 3300 K expand perfectly out from the combustion chamber into air of 1×10^5 Pa, what should be the cross-sectional area at the nozzle exit?

11. When the rocket in Problem 10 flies at an altitude where the pressure is 2×10^4 Pa, what is the obtainable thrust from the rocket?

12. A supersonic flow of Mach 2, pressure 5×10^4 Pa and temperature −15°C develops a normal shock wave. What is the Mach number, flow velocity and pressure behind the wave?